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Compression Bases in Unital Groups

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We introduce and launch a study of compression bases in unital groups. The family of all compressions on a compressible group and the family of all direct compressions on a unital group are examples of compression bases. In this article we show that the properties of the compatibility relation in a compressible group generalize to unital groups with compression bases.

KEY WORDS: normal sub-effect algebra; compatibility; unital group; compression; compressible group; compression base.

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1. NORMAL SUB-EFFECT ALGEBRAS

If E is an effect algebra (Foulis and Bennett, 1994), then a *Mackey decomposition* in E of the ordered pair $(e, f) \in E \times E$ is an ordered triple $(e_1, f_1, d) \in E \times E \times E$ such that $e_1 \perp f_1$, $(e_1 \oplus f_1) \perp d$, $e = e_1 \oplus d$, and $f = f_1 \oplus d$. If there exists a Mackey decomposition in E of $(e, f) \in E \times E$, then e and f are said to be *Mackey compatible* in E.

Definition 1. Let P be a sub-effect algebra of the effect algebra E (Foulis and Bennett, 1994, Definition 2.6). Then P is a *normal* sub-effect algebra of E iff, for all $e, f \in P$, if $(e_1, f_1, d) \in E \times E \times E$ is a Mackey decomposition in E of (e, f), then $d \in P$.

Suppose that E is an effect algebra, P is a sub-effect algebra of E, $e, f \in P$, and $(e_1, f_1, d) \in E \times E \times E$ is a Mackey decomposition of (e, f) in E. Then e and f are Mackey compatible in E, but not necessarily in P. However, if P is a normal sub-effect algebra of E, then $d \in P$ and, since $e_1 \oplus d = e$, $f_1 \oplus d = f$, and $d, e, f \in P$, it follows that $e_1, f_1 \in P$, whence $(e_1, f_1, d) \in P \times P \times P$ is a Mackey decomposition in P of (e, f). Therefore, *if P is a normal sub-effect*

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algebra of E and $e, f \in P$, then e and f are Mackey compatible in E iff e and f are Mackey compatible in P.

Example 1. The center of an effect algebra E (Greechie *et al.*, 1995) is a normal sub-effect algebra of E.

Recall that G is a *unital group* with *unit* u and *unit interval* E iff G is a directed partially ordered abelian group (Goodearl, 1986), such that $u \in G^+ := \{g \in G \mid 0 \le g\}$, $E := \{e \in G \mid 0 \le e \le u\}$, and every element $g \in G^+$ can be written as $g = \Sigma_{i=1}^n e_i$ with $e_i \in E$ for $i = 1, 2, \ldots, n$ (Foulis, 2003, p. 436). (The symbol $:=$ means "equals by definition.")

Suppose that G is a unital group with unit u and unit interval E. Then E is an effect algebra with unit u under the partially defined binary operation $\oplus$ obtained be restriction of $+$ on G to E (Bennett and Foulis, 1997). We note that a sub-effect algebra P of E is a normal sub-effect algebra of E iff, for all $e, f, d \in E$ with $e + f + d \le u$, we have $e + d, f + d \in P \Rightarrow d \in P$.

Example 2. Let $\mathfrak{H}$ be a Hilbert space. Then the additive abelian group $\mathbb{G}$ of all bounded self-adjoint operators on $\mathfrak{H}$, partially ordered in the usual way, is a unital group with unit $\mathbf{1}$. The unit interval $\mathbb{E}$ in $\mathbb{G}$ is the standard effect algebra of all effect operators on $\mathfrak{H}$, and the orthomodular lattice $\mathbb{P}$ of all projection operators on $\mathfrak{H}$ is a normal sub-effect algebra of $\mathbb{E}$.

2. RETRACTIONS AND COMPRESSIONS

Let G be a unital group with unit u and unit interval E. A *retraction* on G with *focus* p is defined to be an order-preserving group endomorphism $J: G \to G$ with $p = J(u) \in E$ such that, for all $e \in E$, $e \le p \Rightarrow J(e) = e$. A retraction J on G with focus p is called a *compression* on G iff $J(e) = 0 \Rightarrow e \le u - p$ holds for all $e \in E$ (Foulis, 2004).

The unital group G always admits at least two compressions, namely the zero mapping, $g \mapsto 0$ for all $g \in G$ and the identity mapping $g \mapsto g$ for all $g \in G$. Suppose J is a retraction with focus p on G. Then, J is idempotent (i.e., $J \circ J = J$) and $J(p) = p$. Also, for all $e \in E$, $e \le u - p \Rightarrow J(e) = 0$ and, if J is a compression, then $e \le u - p \Leftrightarrow J(e) = 0$ (Foulis, 2004).

Lemma 1. *Let G be a unital group with unit u and unit interval E. Suppose that J is a compression on G with focus p, and J' is a retraction on G with focus $u - p$. Then, for all $g \in G^+$, $J(g) = 0 \Leftrightarrow J'(g) = g$.*

Proof: Let $e \in E$. As $0 \le e \le u$, we have $0 \le J'(e) \le J'(u) = u - p$, whence $J(J'(e)) = 0$. Since E generates G as a group and $J \circ J'$ is an endomorphism on G, we have $J(J'(g)) = 0$ for all $g \in G$. As J is a compression with focus p, it follows that $J(e) = 0 \Rightarrow e \leq u - p \Rightarrow J'(e) = e$. Now let $g \in G^+$ and write $g = \Sigma_{i=1}^n e_i$ with $e_i \in E$ for $i = 1, 2, \ldots, n$. If $J(g) = 0$, then $\Sigma_{i=1}^n J(e_i) = 0$ and, since $0 \leq J(e_i)$ for $i = 1, 2, \ldots, n$, it follows that $J(e_i) = 0$ for $i = 1, 2, \ldots, n$, whence $J'(e_i) = e_i$ for $i = 1, 2, \ldots, n$, and therefore $J'(g) = g$. Conversely, if $J'(g) = g$, then $J(g) = J(J'(g)) = 0$. □

A *compressible group* is defined to be a unital group G such that (1) every retraction on G is uniquely determined by its focus, and (2) if J is a retraction on G, there exists a retraction J' on G such that, for all $g \in G^+$, $J(g) = 0 \Leftrightarrow J'(g) = g$ and $J'(g) = 0 \Leftrightarrow J(g) = g$ (Foulis, 2004, Definition 3.3). If G is a compressible group, then an element $p \in G$ is called a *projection* iff it is the focus of a retraction on G. Suppose that G is a compressible group and P is the set of all projections in G. Then every retraction on G is a compression, and if $p \in P$, then the unique retraction (hence compression) on G with focus p is denoted by J_p. The set P is a sub-effect algebra of E and, in its own right, it forms an orthomodular poset (OMP) (Foulis, 2003, Corollary 5.2 (iii)).

Example 3. Let A be a unital C*-algebra and let G be the additive group of all self-adjoint elements in A. Then G is a unital group with unit 1 and positive cone $G^+ = \{aa^* \mid a \in A\}$. The unital group G is a compressible group with $P = \{p \in G \mid p = p^2\}$ and, if $p \in P$, then $J_p(g) = pgp$ for all $g \in G$ (Foulis, 2004).

Theorem 1. *Let G be a compressible group with unit u and unit interval E. Then: (i) P is a normal sub-effect algebra of E. (ii) If $p, q, r \in P$ with $p + q + r \leq u$, then $J_{p+r} \circ J_{q+r} = J_r$.*

Proof: (i) By (Foulis, 2003, Corollary 5.2 (ii)), P is a sub-effect algebra of E. Suppose $e, f, d \in E$, $e + f + d \leq u$, $e + d \in P$, $f + d \in P$, and define $J := J_{e+d} \circ J_{f+d}$. Then $J\colon G \to G$ is an order-preserving endomorphism and $J(u) = J_{e+d}(J_{f+d}(u)) = J_{e+d}(f + d) = J_{e+d}(f) + J_{e+d}(d)$. But, $e + f + d \leq u$, so $f \leq u - (e + d)$, and $d \leq e + d$, whence $J(u) = 0 + d = d$. Suppose $h \in E$ with $h \leq d$. Then $h \leq e + d,\ f + d$, and it follows that $J(h) = J_{e+d}(J_{f+d}(h)) = J_{e+d}(h) = h$. Therefore J is a retraction with focus d, so $d \in P$.

(ii) If $p, q, r \in P$ and $p + q + r \leq u$, then by the proof of (i) above with e replaced by p, f replaced by q, and d replaced by r, we have $J_{p+r} \circ J_{q+r} = J_r$. □

3. COMPRESSION BASES

By Theorem 1, the notion of a "compression base," as per the following definition, generalizes the family $(J_p)_{p \in P}$ of compressions in a compressible group.

Definition 2. Let G be a unital group with unit interval E. A family $(J_p)_{p \in P}$ of compressions on G, indexed by a normal sub-effect algebra P of E, is called a *compression base* for G iff (i) each $p \in P$ is the focus of the corresponding compression J_p, and (ii) if $p, q, r \in P$ and $p + q + r \leq u$, then $J_{p+r} \circ J_{q+r} = J_r$.

The conditions for a unital group to be a compressible group are quite strong and they rule out many otherwise interesting unital groups. On the other hand, the notion of a unital group G with a specified compression base $(J_p)_{p \in P}$ is very general, yet most of the salient properties of projections and compressions for a compressible group generalize, mutatis mutandis, to the elements $p \in P$ and to the compressions J_p in the compression base for G.

Example 4. A retraction J on the unital group G is *direct* iff $J(g) \leq g$ for all $g \in G^+$ (Foulis, 2004, Definition 2.6). For instance, the zero mapping $g \mapsto 0$ and the identity mapping $g \mapsto g$ for all $g \in G$ are direct compressions on G. Let P be the set of all foci of direct retractions on G. Then P is a sub-effect algebra of the center of E. Also, if $p \in P$, there is a unique retraction J_p on G with focus p, and J_p is a compression. Furthermore, the family $(J_p)_{p \in P}$ is a compression base for G.

Standing Assumption. *In the sequel, we assume that G is a unital group with unit u and unit interval E and that $(J_p)_{p \in P}$ is a compression base for G.*

Theorem 2. *P is an orthomodular poset and, if $p \in P$ and $g \in G^+$, then $J_p(g) = 0 \Leftrightarrow J_{u-p}(g) = g$.*

Proof: By (Foulis, 2004, Lemma 2.3 (iv)), every element in P is a principal, hence a sharp, element of E. Therefore, P is an OMP. That $J_p(g) = 0 \Leftrightarrow J_{u-p}(g) = g$ for $p \in P$ and $g \in G^+$ follows from Lemma 1. □

Lemma 2. *If $p, q \in P$, then the following conditions are mutually equivalent: (i) $q \leq p$. (ii) $J_p \circ J_q = J_q$. (iii) $J_p(q) = q$. (iv) $J_q \circ J_p = J_q$. (v) $J_q(p) = q$.*

Proof:

(i) $\Rightarrow$ (ii). Assume (i). Then $p - q \in P$ and $(p - q) + 0 + q = p \leq u$, hence, by Definition 2. (ii), $J_{(p-q)+q} \circ J_{0+q} = J_q$, i.e., $J_p \circ J_q = J_q$.

(ii) $\Rightarrow$ (iii). Assume (ii). Then $J_p(q) = J_p(J_q(u)) = J_q(u) = q$.

(iii) $\Rightarrow$ (iv). Assume (iii). Then $q = J_p(q) \leq p$. Therefore $p - q \in P$ and $0 + (p - q) + q = p \leq u$; hence, by Definition 2. (ii), $J_{0+q} \circ J_{(p-q)+q} = J_q$, i.e., $J_q \circ J_p = J_q$.

(iv) $\Rightarrow$ (v). Assume (iv). Then $q = J_q(u) = J_q(J_p(u)) = J_q(p)$.

(v) $\Rightarrow$ (i). Assume (v). Then $J_q(u - p) = q - q = 0$, so $u - p = J_{u-q}(u - p) = J_{u-q}(u) - J_{u-q}(p) = u - q - J_{u-q}(p)$, i.e., $q + J_{u-q}(p) = p$. But $0 \leq J_{u-q}(p)$, so $q \leq p$. □

4. COMPATIBILITY

We maintain our standing assumption that $(J_p)_{p\in P}$ is a compression base for the unital group G with unit u and unit interval E. The notion of compatibility in a compressible group (Foulis, 2003, Definition 4.1) carries over, as follows, to G.

Definition 3. If $p \in P$, we define $C(p) := \{g \in G \mid g = J_p(g) + J_{u-p}(g)\}$. If $g \in C(p)$, we say that g is *compatible* with $p \in P$. For $p, q \in P$, we often write the condition $q \in C(p)$ in the alternative form qCp.

We devote the remainder of this article to showing that *the fundamental properties of compatibility in a compressible group generalize to a unital group with a compression base.*

Lemma 3. *Let $p \in P$ and $g \in G$. Then $J_p(g) \leq g \Rightarrow g \in C(p)$, and $0 \leq g \in C(p) \Rightarrow J_p(g) \leq g$.*

Proof: Suppose $J_p(g) \leq g$. Then $0 \leq g - J_p(g)$ and $J_p(g - J_p(g)) = J_p(g) - J_p(g) = 0$, whence $g - J_g(g) = J_{u-p}(g - J_p(g)) = J_{u-p}(g) - 0 = J_{u-p}(g)$, i.e., $g = J_p(g) + J_{u-p}(g)$, and therefore, $g \in C(p)$. Conversely, if $0 \leq g \in C(p)$, then $0 \leq J_{u-g}(g)$, whence $J_p(g) \leq J_p(g) + J_{u-p}(g) = g$. □

Theorem 3. *Let $p, q \in P$. Then the following conditions are mutually equivalent: (i) $J_p \circ J_q = J_q \circ J_p$. (ii) $J_p(q) = J_q(p)$. (iii) $J_p(q) \leq q$. (iv) p is Mackey compatible with q in E. (v) p is Mackey compatible with q in P. (vi) $\exists r \in P, J_p \circ J_q = J_r$. (vii) $J_p(q) \in P$. (viii) qCp.*

Proof:

(i) $\Rightarrow$ (ii). If (i) holds, then $J_p(q) = J_p(J_q(u)) = J_q(J_p(u)) = J_q(p)$.

(ii) $\Rightarrow$ (iii). If (ii) holds, then $J_p(q) = J_q(p) \leq q$.

(iii) $\Rightarrow$ (iv). Let $r := J_p(q)$ and assume that $r \leq q$. Then $0 \leq r \leq p, q$, whence $e := p - r \in E$ and $f := q - r \in E$ with $e + r = p$ and $f + r = q$. As $J_p(f) = J_p(q - r) = r - r = 0$, we have $f \leq u - p$, whence $e + f + r = f + p \leq u$, and it follows the p is Mackey compatible with q in E.

(iv) $\Rightarrow$ (v). As P is a normal sub-effect algebra of E, we have (iv) $\Rightarrow$ (v).

(v) $\Rightarrow$ (vi). If (v) holds, there exist $e, f, r \in P$ with $e + f + r \leq u$, $p = e + r$ and $q = f + r$. Therefore, by Definition 2. (ii), $J_p \circ J_q = J_{e+r} \circ J_{f+r} = J_r$.

(vi) $\Rightarrow$ (vii). Suppose that $r \in P$ and $J_p \circ J_q = J_r$. Then $J_p(q) = J_p(J_q(u)) = J_r(u) = r \in P$.

(vii) $\Rightarrow$ (viii) Assume (vii) and let $r := J_p(q) \in P$. Then $J_r(q) \leq r \leq p$, so $0 \leq r - J_r(q)$. Thus, by Lemma 2, $r - J_r(q) = r - (J_r \circ J_p)(q) = r - J_r(J_p(q)) = r - J_r(r) = r - r = 0$, i.e., $r = J_r(q)$.

Therefore, $J_r(u - q) = r - r = 0$, so $u - q \le u - r$, i.e., $r \le q$, and it follows from Lemma 3 that pCq.

(viii) $\Rightarrow$ (i). Assume that qCp. Then, by Lemma 3, $J_p(q) \le q$, so (iii) holds. We have already shown that (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), so there exist $e, f, r \in P$ with $e + f + r \le u$, $p = e + r$, and $q = f + r$. Therefore, by Definition 2. (ii), $J_p \circ J_q = J_{e+r} \circ J_{f+r} = J_r = J_{f+r} \circ J_{e+r} = J_q \circ J_p$. □

Because conditions (i), (ii), (iv), and (v) in Theorem 3 are symmetric in p and q, so are conditions (iii), (vi), (vii), and (viii). In particular, for $p, q \in P$, we have $pCq \Leftrightarrow qCp$.

Corollary 1. *Let $p, q \in P$ and suppose that pCq. Then $J_q(p) = J_p(q) = p \wedge q$ is the greatest lower bound of p and q both in E and in P, and $J_p \circ J_q = J_q \circ J_p = J_{p \wedge q}$.*

Proof: Suppose that $p, q \in P$ and pCq. By Theorem 3, there exists $r \in P$ with $J_p \circ J_q = J_q \circ J_p = J_r$. Thus, $r = J_p(J_q(u)) = J_p(q) = J_q(p) \le p, q$. If $e \in E$ with $e \le p, q$, then $e = J_p(J_q(e)) = J_r(e) \le r$, so r is the greatest lower bound of p and q in E, hence also in P. □

Theorem 4. *Let $v \in P$ and define $H := J_v(G)$, $E_H := \{e \in E \mid e \le v\}$, and $P_H := \{q \in P \mid q \le v\}$. For each $q \in P_H$, let J_q^H be the restriction of J_q to H. Then: (i) With the induced partial order, $H = \{h \in G \mid h = J_v(h)\}$ is a unital group with unit v and unit interval $H \cap E = E_H$. (ii) $H \cap P = P_H$, and if $q \in P_H$, then J_q^H is a compression on H. (iii) P_H is a normal sub-effect algebra of E_H. (iv) $(J_q^H)_{q \in P_H}$ is a compression base for H.*

Proof:

(i) By (Foulis, 2003, Lemma 2.4), H is a unital group with unit v and unit interval $H \cap E$. As J_v is idempotent, $H = \{h \in G \mid h = J_v(h)\}$. Thus, for $e \in E$, $e \le v \Leftrightarrow e = J_v(e) \Leftrightarrow e \in H$, whence $H \cap E = \{e \in E \mid e \le v\}$.

(ii) As $P \subseteq E$, we have $H \cap P = P_H$. If $h \in H$ and $q \in P_H$, then by Lemma 2, $J_q(h) = J_v(J_q(h)) \in H$. Therefore $J_q^H \colon H \to H$ is an order-preserving group endomorphism, and by Lemma 2 again, $J_q^H(v) = J_q(v) = q$. Also, if $e \in E_H$ with $e \le q$, then $J_q^H(e) = J_q(e) = e$, so J_q^H is a retraction on H. Suppose $e \in E_H$ and $J_q^H(e) = 0$. Then $e \le u - q$, so $e + q \le u$. By (Foulis, 2004, Lemma 2.3 (iv)), v is a principal element of E, hence, since $0 \le e, q \le v$, it follows that $e + q \leq v$, i.e., $e \leq v - q$. Hence, J_q^H is a compression on H.

(iii) Suppose $e, f, d \in E_H$, $e + f + d \leq v$, and $e + d, f + d \in P_H$. Then $e, f, d \in E$, $e + f + d \leq v \leq u$, and $e + d, f + d \in P$. As P is a normal sub-effect algebra of E, it follows that $d \in P$. But $d \leq v$, so $d \in P_H$.

(iv) Suppose $s, t, r \in P_H$ with $s + t + r \leq v$. Then $s, t, r \in P$ with $s + t + r \leq u$, whence $J_{s+r} \circ J_{t+r} = J_r$, and it follows that $J_{s+r}^H \circ J_{t+r}^H = J_r^H$. □

Theorem 5. *Let $v \in P$ and define $C := C(v)$. For each $s \in C \cap P$, let J_s^C be the restriction of J_s to C. Then: (i) With the induced partial order, C is a unital group with unit u and unit interval $C \cap E = \{e + f \mid e, f \in E, e \leq v, f \leq u - v\}$. (ii) If $s \in C \cap P$, then J_s^C is a compression on C. (iii) $C \cap P$ is a normal sub-effect algebra of $C \cap E$. (iv) $(J_s^C)_{s \in C \cap P}$ is a compression base for C.*

Proof: Part (i) follows from (Foulis, 2003, Lemma 4.2 (iv)), part (iii) is obvious, and part (iv) is easily confirmed once part (ii) is proved. To prove part (ii), assume that $g \in C = C(v)$ and $s \in P \cap C$. Then, by Lemma 2, $J_s^C(g) = J_s(J_v(g) + J_{u-v}(g)) = J_s(J_v(g)) + J_s(J_{u-v}(g)) = J_v(J_s(g)) + J_{u-v}(J_s(g))$, so $J_s^C(g) = J_s(g) \in C(v) = C$. Therefore $J_s^C: C \to C$ is an order-preserving group endomorphism, hence it is obviously a compression on C. □

Definition 4. If C and W are unital groups with units u and w, respectively, and if $(J_q^C)_{q \in Q}$ and $(J_t^W)_{t \in T}$ are compression bases in C and W, respectively, then an order-preserving group homomorphism $\phi: C \to W$ is called a *morphism of unital groups with compression bases* iff $\phi(u) = w$, $\phi(Q) \subseteq T$, and $J_{\phi(q)}^W \circ \phi = \phi \circ J_q^C$ for all $q \in Q$.

We omit the straightforward proof of the following theorem.

Theorem 6. *Suppose $v \in P$ and define $H := J_v(G)$, $K := J_{u-v}(G)$, and $C := C(v)$. Organize H, K, and C into unital groups with compression bases $(J_q^H)_{q \in P_H}$, $(J_r^K)_{r \in P_K}$, and $(J_s^C)_{s \in C \cap P}$, respectively, as in Theorems 4 and 5. Let η be the restriction to C of J_v and let κ be the restriction to C of J_{u-v}. Then $\eta: C \to H$ and $\kappa: C \to K$ are surjective morphisms of unital groups with compression bases and, in the category of unital groups with compression bases, η and κ provide a representation of C as a direct product of H and K.*

In subsequent papers we shall prove that all of the major results in (Foulis, 2003, 2004, 2005) can be generalized to unital groups with compression bases.

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